Differential Equations on a Network: from Dynamics to Structure

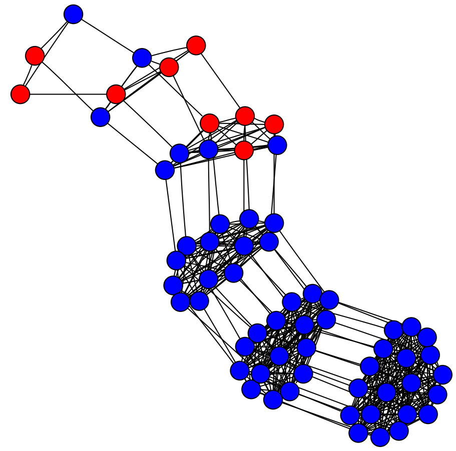
Katy Bold, kbold@princeton.edu
Aric Hagberg, hagberg@lanl.gov
Pieter Swart, swart@lanl.gov

Networks are used in many different fields to model interactions between objects (people, computers, etc.). The structure of a network is encoded in the connections (edges, links, bonds) between nodes (vertices, sites, actors), and understanding network structure provides insight into network function. For example, in epidemiology, a vaccination program can be guided by identifying the people who interact the most with others (the super-spreaders). In distributed computing, network structure plays a key role in designing an optimal partition of a parallel process. In many biological and infrastructure networks, robustness appears to be closely connected with network structure.

We study network structure by using the dynamics of differential equations to identify key structural features. For classical partial differential equations, it is well known that domain geometry (for example, shape and dimension) greatly affects the nature of solutions. The same is true for networks, and we use this to characterize various network properties.

For example, the dynamics of a bistable reaction-diffusion equation on a network allow one to distinguish between small groups of highly connected nodes and larger groups of highly connected nodes. The resulting evolution is a competition between two terms: a diffusion term driving the solution to uniformity, and a reaction term driving the solution into one of two (or more) states. Connections between nodes determine how diffusion affects the dynamics, and highly connected clusters of nodes evolve to the same state, as shown in the figure.

We chose differential equations with interesting and well-studied solutions in the continuum case, and where interface dynamics are determined by mean curvature. On a network, the motion of interfaces (or lack thereof) is used to identify structure of the network. In contrast to the continuum dynamics, the inherent discreteness of networks can cause sharp interfaces to lock up. We compare classical theoretical results with numerical results in the discrete case.



The extraction of network structure using dynamics is displayed on a simple network made from connected cliques — collections of nodes with every pair connected. The value at each node is governed by a reaction-diffusion equation, and the color of the vertex shows the sign (positive or negative) of the value at equilibrium. The dynamics are different on larger cliques (uniform solution) than on smaller cliques (nonuniform solution). This property can be used to find communities, modular units, or clusters in arbitrary networks.

This work provides fundamental insights into how network structure affects network dynamics and suggests novel algorithms to compute structural properties of networks.

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